# Towards the conservation laws and Lie symmetries for the Khokhlov-Zabolotskaya equation in three dimensions



1986 J. Phys. A: Math. Gen. 19 1775

(http://iopscience.iop.org/0305-4470/19/10/018)

J. Phys. A: Math. Gen. 19 (1986) 1775-1781. Printed in Great Britain

# Towards the conservation laws and Lie symmetries for the Khokhlov–Zabolotskaya equation in three dimensions

A Roy Chowdhury and Minati Nasker

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta 700 032, India

Received 25 February 1985

**Abstract.** We have obtained the explicit structure for the generating function of Lie symmetries for the Khokhlov-Zabolotskaya equation describing the propagation of a sound beam in a non-linear medium. In the absence of a canonical framework we have derived the set of conservation laws through the technique of differential forms and prolongation. Such conservation laws are of utmost importance for the analysis of a sound beam in the medium.

The equation describing the propagation of a sound beam in a non-linear medium is [1]

$$\rho_{xt}-(\rho\rho_x)_x=\rho_{yy} \tag{1}$$

in two space and one time dimensions. This, or a similar equation in three space and one time dimensions, is known as the Khokhlov-Zabolotskaya equation. We have obtained here the Lie point symmetries and the conservation laws associated with equation (1). Since it is known that a direct relation between the symmetries and conservation laws exists for equations deducible from a variational principle, we have followed here an elegant approach of differential forms and closure, due to Walhquist and Eastabrook [6] to obtain the structure of conserved quantities.

Let us consider a set of transformations for equation (1) in the form

$$\begin{aligned}
&\rho\to\rho+\varepsilon\eta_1(\rho,x,t,y)=\rho^*\\
&x\to x+\varepsilon\eta_2(\rho,x,t,y)=x^*\\
&t\to t+\varepsilon\eta_3(\rho,x,t,y)=t^*\\
&y\to y+\varepsilon\eta_4(\rho,x,t,y)=y^*
\end{aligned} \tag{2}$$

and demand that equation (1) remains invariant under (2), that is

$$\rho^*_{t^*t^*}-(\rho^*\rho^*_{x^*})_{x^*}=\rho^*_{y^*y^*}. \tag{3}$$

By repeated use of the chain rule of differentiation we can obtain the transformation rules for $\rho_x$, $\rho_y$, $\rho_{xx}$, etc, and write them as [2]

$$\rho^*_{x^*}\to\rho_x+D^x\eta_1\qquad\rho^*_{y^*}\to\rho_y+D^y\eta_2 \tag{4}$$

and so on, where only terms of the order of $\varepsilon$ have been retained, and $D^x$ and $D^y$ denote respectively the total derivative evaluated according to the following formulae:

$$\begin{aligned}
D^x\eta_1&=\frac{\partial\eta_1}{\partial x}+\frac{\partial\eta_1}{\partial\rho}\cdot\rho_x-\sum_2^4\frac{\partial\eta_k}{\partial x}\rho_k-\sum_2^4\frac{\partial\eta_k}{\partial\rho}\rho_x\rho_k\\
D^y\eta_1&=\frac{\partial\eta_1}{\partial y}+\frac{\partial\eta_1}{\partial\rho}\rho_y-\sum_2^4\frac{\partial\eta_k}{\partial y}\rho_k-\sum_2^4\frac{\partial\eta_k}{\partial\rho}\rho_y\rho_k.
\end{aligned} \tag{5}$$

  

Similar expressions for higher-order derivatives can be constructed (some of them are also given in reference [2]). Substituting expressions like (4) and (5) in (3) and equating like powers of $\rho_x$, $\rho_y$, $\rho_y^2$, $\rho_{xy}$, etc, after taking care of the original equation (1) we obtain the following equations for the determination of the transformation functions. (Here we have used the notation that $\rho_k = \partial\rho/\partial x_k$, $Z_1 = x$, $Z_2 = t$, $Z_3 = y$.)

$$
\begin{gathered}
\frac{\partial^2\eta_1}{\partial Z_2\,\partial\rho} - \frac{\partial^2\eta_2}{\partial Z_1 \partial Z_2} - \rho\left(2\frac{\partial^2\eta_1}{\partial Z_1\,\partial\rho} - \frac{\partial^2\eta_2}{\partial Z_1^2}\right) - 2\frac{\partial\eta_1}{\partial Z_1} + \frac{\partial^2\eta_2}{\partial Z_3^2} = 0 \\
-\frac{\partial^2\eta_4}{\partial Z_1\,\partial Z_2} + \rho\frac{\partial^2\eta_4}{\partial Z_1^2} - \left(2\frac{\partial^2\eta_1}{\partial Z_3\,\partial\rho} - \frac{\partial^2\eta_4}{\partial Z_3^2}\right) = 0 \\
\frac{\partial^2\eta_1}{\partial\rho_2} - 2\frac{\partial^2\eta_4}{\partial Z_3\,\partial\rho} = 0 \qquad -\frac{\partial\eta_4}{\partial Z_2} + 2\rho\frac{\partial\eta_4}{\partial Z_1} + 2\frac{\partial\eta_2}{\partial Z_3} = 0 \\
\frac{\partial^2\eta_4}{\partial\rho^2} = 0 \qquad -\frac{\partial\eta_3}{\partial Z_1} = 0 \qquad \frac{\partial^2\eta_3}{\partial\rho^2} = 0 \qquad \frac{\partial^2\eta_2}{\partial\rho^2} = 0 \\
-\frac{\partial\eta_4}{\partial Z_1} + 2\frac{\partial\eta_3}{\partial Z_3} = 0 \qquad \frac{\partial^2\eta_1}{\partial\rho^2} - 2\frac{\partial^2\eta_4}{\partial Z_3\,\partial\rho} = 0.
\end{gathered}
\tag{6}
$$

These sets of equations can be solved effectively to yield

$$
\begin{aligned}
\eta_1 &= (a-b)\rho + cy + d \\
\eta_2 &= ax - (cy+d) + e \\
\eta_3 &= bt + e \\
\eta_4 &= \tfrac{1}{2}(a+b)y - ct^2 + e.
\end{aligned}
\tag{7}
$$

The Lagrange equations pertaining to (7) can be written as

$$
\frac{\mathrm{d}\rho}{\eta_1} = \frac{\mathrm{d}x}{\eta_2} = \frac{\mathrm{d}y}{\eta_4} = \frac{\mathrm{d}t}{\eta_3}. \tag{8}
$$

Integrating (8) we obtain the following similarity structure

$$
\begin{aligned}
X &= xt^{-p} + lyt^{1-p} + \tfrac{1}{2}l^2t^{3-p} \\
Y &= yt^{-(p+1)/2} + lt^{(3-p)/2} \\
\rho(x, y, t) &= ly + \tfrac{1}{2}l^2t^2 + t^{p-1}\phi_1(X, Y)
\end{aligned}
\tag{9}
$$

$\phi_1$ being an arbitrary function of two variables. When we plug this expression for $\rho(X, Y, t)$ in the original equation (1) we obtain the following equation for $\phi_1$ in two dimensions

$$
\phi_{1X}^2 + \phi_{1X} + \phi_1\phi_{1XX} + \tfrac{1}{2}(p+1)Y\phi_{1XY} + pX\phi_{1XX} + \phi_{1YY} = 0 \tag{10}
$$

with $p = a/b$, $l = 2c/(3b-a)$.

Since equation (10) is still a partial differential equation it can again admit a further set of point transformations

$$
\begin{aligned}
\phi_1 &\to \phi_1^* = \phi_1 + \varepsilon\xi_1(\phi_1, X, Y) \\
X &\to X^* = X + \varepsilon\xi_2(\phi_1, X, Y) \\
Y &\to Y^* = Y + \varepsilon\xi_3(\phi_1, X, Y)
\end{aligned}
\tag{11}
$$

along with

$$\xi_1 = \frac{b-2a}{7b}\phi_1 \qquad \xi_2 = \frac{b-2a}{7b}X \qquad \xi_3 = \frac{b-2a}{14b}Y \tag{12}$$

which is nothing but a simple scaling transformation. Integrating the corresponding Lagrange equations, we obtain

$$\phi_1 = Y^2\tilde{\phi}(Z) = Y^2\tilde{\phi}(X/Y^2) \tag{13}$$

whence equation (10) is reduced to a non-linear ordinary differential equation

$$\tilde{\phi}_Z^2 + (1-2Z)\tilde{\phi}_Z + (\phi - Z + 4Z^2)\tilde{\phi}_{ZZ} + 2\tilde{\phi}_Z = 0. \tag{14}$$

However, it is unfortunate that this equation is not mentioned in the Painlevé classification mentioned in Ince [4]. So it is not very clear as to how one can obtain an explicit solution of (14) and thereby that of the original problem. Of course one can make a critical point analysis following Ince or Ablowitz *et al* to ascertain whether (14) is integrable or not, but such an analysis is outside the scope of this paper. So we observe that the Lie symmetry generator in the first stage of reduction is of the form

$$\eta = (a-b)\rho + cy + d + [ax - (cy+d) + e]\rho_x + \tfrac{1}{2}(a+b)y - ct^2 + e]\rho_y + (bt+e)\rho_t. \tag{15}$$

The relevant operators can be put in the form [5]:

$$\begin{aligned}
X_1 &= \rho\frac{\partial}{\partial\rho} + x\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial y} \\
X_2 &= y\left(\frac{\partial}{\partial\rho} - \frac{\partial}{\partial x}\right) - t^2\frac{\partial}{\partial y} \\
X_3 &= -\frac{\partial}{\partial\rho} + t\frac{\partial}{\partial t} + \frac{y}{2}\frac{\partial}{\partial y} \\
X_4 &= \frac{\partial}{\partial\rho} - \frac{\partial}{\partial x} \\
X_5 &= \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{\partial}{\partial y}.
\end{aligned} \tag{16}$$

It is not very difficult to see that these operators form a Lie algebra which operates on the solution manifold of equation (1). Similar considerations are also valid for the generator $\xi$ at the second stage of reduction.

Since the existence of symmetries indicates that there should exist some non-trivial conservation laws, here we try to extract the structure of such conserved quantities from a general formalism rather than from the symmetries deduced above. The main reason for such a treatment is that for systems not known to possess a Lagrangian or Hamiltonian structure it is not possible to make a one-to-one connection between the symmetries and conservation laws. It is known that these conservation laws are very important for the analysis of the dispersion of the sound beam during the course of its propagation. As an alternative route to the conservation laws we here follow an effective as well as an elegant method based on the use of differential form, prolongation and closure under exterior differentiation [6]. This method is perhaps unique in the sense that it can be used without any modification in any number of dimension, specially in three dimensions, in our case.

To write our equation in the language of differential form we put [7]

$$p=\rho_x \qquad q=\rho_y \tag{17}$$

so that the equation becomes

$$p_t-p^2-\rho p_x-q_y=0. \tag{18}$$

Now, it is easy to observe that our original non-linear equation (1) is equivalent to the following set of differential forms when sectioned in a proper fashion:

$$\begin{aligned}
\alpha_1&=p\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}\rho\wedge\mathrm{d}t\wedge\mathrm{d}y\\
\alpha_2&=q\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}\rho\wedge\mathrm{d}x\wedge\mathrm{d}t\\
\alpha_3&=\mathrm{d}p\wedge\mathrm{d}y\wedge\mathrm{d}x-p^2\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\rho\,\mathrm{d}p\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}q\wedge\mathrm{d}x\wedge\mathrm{d}t.
\end{aligned} \tag{19}$$

These sets of differential forms which are equivalent to the original differential equation under proper sectioning can generate the conservation laws associated with the system if we follow the reasoning given below. Finding a conservation law is equivalent to searching for a 2-form:

$$w=F\,\mathrm{d}x\wedge\mathrm{d}t+G\,\mathrm{d}t\wedge\mathrm{d}y+H\,\mathrm{d}y\wedge\,\mathrm{d}x \tag{20}$$

such that the exterior derivatives of $w$, i.e.

$$\mathrm{d}w=\left(\frac{\partial f}{\partial y}+\frac{\partial G}{\partial x}+\frac{\partial H}{\partial t}\right)\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y \tag{21}$$

remains in the ideal generated by the set of closed forms $\alpha_i$ $(i=1,2,3)$. In the mathematical language we demand

$$\mathrm{d}w=\sum f_i\alpha_i. \tag{22}$$

Equation (22) yields some differential equation for $F$ and $G$ which upon solution yields a set of charge and current in accordance with (21). An interesting observation at this point is that a new and non-trivial set of conservation laws can be generated if the basic set of variables (called independent variables in the language of prolongation theory) $p$ and $q$ are extended to include higher-order derivatives such as $\rho_{xx}$ and $\rho_{yy}$, etc, and to include in equation (21) all other differential forms which are equivalent to the derived consequence of the original equation. In our calculation elaborated below we illustrate this in detail. However at this stage an important observation may be made. For some combinations of the basis variables spanning the jet space the solution structure of $F$ and $G$ may not be completely new but a simple consequence of those previously obtained.

(i) Our equation (1) is itself in the form of a conservation law. To generate a new one let us consider equation (1) and the derived equation obtained by differentiating with respect to $x$, which is

$$\rho_{xxt}-3\rho_x\rho_{xx}-\rho\rho_{xxx}-\rho_{yyx}=0. \tag{22a}$$

Let us define the basis variables

$$p=\rho_x \qquad q=\rho_y \qquad S=\rho_{xx} \qquad Z=\rho_{xy}=p_y$$

and consider the problem of obtaining $F$ and $G$ as functions of $(p, q, S, Z, \rho)$ so that equation (21) is satisfied. In this case we obtain an equation of the form for the determination of $F$, $G$ and $H$

$$G_S = -\rho H_S \qquad H_S = -F_Z$$

and

$$ZF_p + pG_\rho + p^2 Hp + 3pSH_S - S(\rho H_p + G_p) = 0 \tag{23}$$

when the fundamental set of forms is

$$\begin{aligned}
\alpha_1 &= Z\, dx \wedge dt \wedge dy - dp \wedge dx \wedge dt \\
\alpha_2 &= p\, dx \wedge dt \wedge dy - d\rho \wedge dt \wedge dy \\
\alpha_3 &= dp \wedge dy \wedge dx - p^2\, dx \wedge dt \wedge dy - \rho\, dp \wedge dt \wedge dy - dq \wedge dx \wedge dt \\
\alpha_4 &= ds \wedge dy \wedge dx - 3ps\, dx \wedge dt \wedge dy - \rho\, ds \wedge dt \wedge dy - dZ \wedge dx \wedge dt \\
\alpha_5 &= S\, dx \wedge dt \wedge dy - dp \wedge dt \wedge dy.
\end{aligned} \tag{24}$$

It is then not very difficult to observe that a solution to equation (23) is

$$G = -\alpha\rho S - \alpha p^2 \qquad H = \alpha S \qquad F = -\alpha Z \tag{25}$$

which is seen to satisfy equation (20) on the solution manifold of equation (1).

(ii) Let us now change the basis variables to the following set

$$p = \rho_2 \qquad q = \rho_y \qquad Z = \rho_{xy} \qquad S = \rho_{xx} = p_x \qquad \gamma = \rho_{yy} = q_y$$

and consider equation (1) with the one obtained by taking a derivative of (1) which is

$$\rho_{xyt} - 2\rho_x\rho_{xy} - \rho_y\rho_{xx} - \rho\rho_{xxy} - \rho_{yyy} = 0. \tag{26}$$

The basis 3-forms are

$$\begin{aligned}
\alpha_1 &= Z\, dx \wedge dt \wedge dy - dq \wedge dt \wedge dy \\
\alpha_2 &= q\, dx \wedge dt \wedge dy - d\rho \wedge dx \wedge dt \\
\alpha_3 &= p\, dx \wedge dt \wedge dy - d\rho \wedge dt \wedge dy \\
\alpha_4 &= \gamma\, dx \wedge dt \wedge dy - dq \wedge dx \wedge dt \\
\alpha_5 &= dZ \wedge dy \wedge dx - 2pZ\, dx \wedge dt \wedge dy - qS\, dx \wedge dt \wedge dy \\
&\qquad - \rho\, dZ \wedge dt \wedge dy - d\gamma \wedge dx \wedge dt \\
\alpha_6 &= S\, dx \wedge dt \wedge dy - dp \wedge dt \wedge dy
\end{aligned} \tag{27}$$

which along with (22) yields

$$\begin{aligned}
&G_Z = -\rho H_Z = \rho F_\gamma \\
&GqZ + qF_\rho + \rho G_\rho + \gamma F_q + 2pZH_Z + SG_p + qSH_Z = 0.
\end{aligned} \tag{28}$$

A non-trivial solution is given by

$$F = -\alpha\gamma \qquad G = -\alpha\rho Z - \alpha pq \qquad H = \alpha Z. \tag{29}$$

(iii) When the set of basis variables is taken to be $Z=\rho_{yy}$, $p=\rho_x$, $q=\rho_y$, $S=\rho_{xx}=p_x$ in (22), the basis 3-forms are

$$\begin{aligned}
\alpha_1 &= Z\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}q\wedge\mathrm{d}x\wedge\mathrm{d}t\\
\alpha_2 &= p\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}\rho\wedge\mathrm{d}t\wedge\mathrm{d}y\\
\alpha_3 &= q\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}\rho\wedge\mathrm{d}x\wedge\mathrm{d}t\\
\alpha_4 &= \mathrm{d}s\wedge\mathrm{d}y\wedge\mathrm{d}x-3pS\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\rho\,\mathrm{d}S\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}Z\wedge\mathrm{d}t\wedge\mathrm{d}y\\
\alpha_5 &= S\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}p\wedge\mathrm{d}t\wedge\mathrm{d}y.
\end{aligned}\tag{30}$$

The corresponding solution obtained is

$$F=\alpha \qquad H=\alpha S \qquad G=-\alpha\rho S-\alpha Z-\alpha p^2. \tag{31}$$

(iv) Similar calculations can also be done by considering the '$t$' derivative of equation (1):

$$\rho_{xtt}-2\rho_x\rho_{xt}-\rho_t\rho_{xx}-\rho\rho_{xxt}-\rho_{yyt}=0$$

and choosing the basic variables as

$$Z=\rho_{xt}=S_x \qquad p=\rho_x \qquad q=\rho_y \qquad r=\rho_{yy}=q_y \qquad S=\rho_t$$

which yields the corresponding differential 3-forms

$$\begin{aligned}
\alpha_1 &= Z\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}s\wedge\mathrm{d}t\wedge\mathrm{d}y\\
\alpha_2 &= p\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}\rho\wedge\mathrm{d}t\wedge\mathrm{d}y\\
\alpha_3 &= q\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}\rho\wedge\mathrm{d}x\wedge\mathrm{d}t\\
\alpha_4 &= \gamma\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}q\wedge\mathrm{d}x\wedge\mathrm{d}t\\
\alpha_5 &= S\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}\rho\wedge\mathrm{d}y\wedge\mathrm{d}x\\
\alpha_6 &= \mathrm{d}Z\wedge\mathrm{d}y\wedge\mathrm{d}x-2pZ\,\mathrm{d}x\wedge\mathrm{d}t\wedge\mathrm{d}y-S\,\mathrm{d}\rho\wedge\mathrm{d}t\wedge\mathrm{d}y\\
&\quad -\rho\,\mathrm{d}Z\wedge\mathrm{d}t\wedge\mathrm{d}y-\mathrm{d}\gamma\wedge\mathrm{d}y\wedge\mathrm{d}x
\end{aligned}\tag{32}$$

giving the following equations for $F$, $G$ and H

$$\begin{gathered}
H_Z=-H_\gamma \qquad G_Z=\rho H_\gamma \qquad G_p=-SH_Z\\
ZG_S+pG_\rho+qF_\rho+\gamma F_q+SH_\rho+2pZH_Z=0
\end{gathered}\tag{33}$$

whose solution is seen to be

$$F=\alpha \qquad G=\alpha\rho Z+\alpha pS \qquad H=\alpha\gamma-\alpha Z. \tag{34}$$

(v) Lastly we mention the case for the set of variables

$$Z=\bar{\rho}_{xy} \qquad p=\rho_x \qquad q=\rho_y \qquad \gamma=\rho_{yy} \qquad S=\rho_{xx}.$$

Without going into the details which are similar to above we can report the following form of $F$, $G$ and $H$:

$$F=-\alpha\gamma-\alpha\rho S-p_\alpha^2 \qquad G=\alpha \qquad H=\alpha Z. \tag{35}$$

In our above computations we have reported a detailed investigation about the symmetry generators and conservation laws of the Khokhlov-Zabolotskaya equation in three dimensions. Since until now only a few equations in three dimensions are known to be admissible in the formalism of a Lax pair, and our present equation does not belong to this class, the method adopted seems to be the only logical step in providing the integrals of motion. It has already been observed that such integrals are useful in determining the extent of broadening of the sound beam in the course of its propagation through the non-linear medium. Further analysis of the actual physical problem employing these results will be the subject matter of a future publication.

## References

[1] Rudenko O V and Soluyan S I 1975 *The Theoretical Basics of Non Linear Acoustics* (Moscow: Nauka)
[2] Bluman G W and Cole I D 1970 *Similarity Methods in Partial Differential Equations* (Berlin: Springer)
[3] Rund H 1976 *Lecture Notes in Physics* **515** (Berlin: Springer)
[4] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
[5] Ovsyannikov L V 1978 *Group Analysis of Differential Equations* (Moscow: Nauka)
[6] Walhquist H D and Eastabrook F B 1975 *J. Math. Phys.* **16** 1
Eastabrook F B and Walhquist H D 1976 *J. Math. Phys.* **17** 1293
[7] Flanders H 1963 *Differential Forms* (New York: Academic)